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The theory of noncommutative dynamical entropy and quantum symbolic dynamics for quantum dynamical systems is analised from the point of view of quantum information theory. Using a general quantum dynamical system as a communication channel one can define different classical capacities depending on the character of resources applied for encoding and decoding procedures and on the type of information sources. It is shown that for Bernoulli sources the entanglement-assisted classical capacity, which is the largest one, is bounded from above by the quantum dynamical entropy defined in terms of operational partitions of unity. Stronger results are proved for the particular class of quantum dynamical systems – quantum Bernoulli shifts. Different classical capacities are exactly computed and the entanglement-assisted one is equal to the dynamical entropy in this case.

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I. INTRODUCTION

The relations between the classical theory of dynamical systems and the theory of classical communication channels are given by the Kolmogorov-Sinai construction of symbolic dynamics and dynamical entropy (K-S entropy)[1]. One can expect that in the quantum domain the interrelations with information theory should be much deeper. This is due to the fact that the quantum theory is a genuine statistical and operational one and the most fundamental process - state preparation followed by measurement - possesses non-trivial information-theoretical meaning. Indeed, take a tunable device which prepars a quantum system in one of the states $\{\psi_1, \psi_2, ...\psi_m\}$ and use another apparatus to perform a measurement of the observable with possible values $\{a_1, a_2, ...a_n\}$. This can be seen as a single operation of an information channel with possible inputs $\{1, 2, ..., m\}$ and outputs $\{1, 2, ..., n\}$. Quantum theory gives statistical predictions about the value of an output provided an input is given and optimalization of the transmitted information is a fundamental physical question.

In the last decade a considerable progress in the theory of quantum communication channels has been achieved [2,3]. However, most of the attention was concentrated on *memoryless noisy channels*. They simulate physical systems essentially composed of noninteracting subsystems (particles) with quantum noise acting independently on each of them. As a consequence, features of dynamics of the information carrier do not enter manifestly the game. In order to investigate more complicated models of communication channels we use a different setting for sending classical information via quantum dynamical systems proposed in [4]. In particular we expect relations between the speed of information transmission through a channel and its chaotic properties characterized by quantum generalizations of *K-S entropy*. In this scheme, not a presence of noise, but the way the perturbations of an initial state of the system propagate determines the efficiency of information processing. It is a well known fact used in modern control systems (e.g. aviation technology) that working in unstable (chaotic) regime is more efficient that in a stable one. A similar phenomenon should be visible in the quantum domain also.

In the theory of communication channels we are interested in asymptotic results valid in the limit of infinitely long messages. Therefore the convenient mathematical description involves infinite quantum systems similar to systems in thermodynamic limit considered in statistical mechanics and quantum field theory. The corresponding mathematical formalism is C^*-algebraic approach [5].

II. QUANTUM DYNAMICAL SYSTEMS IN ALGEBRAIC SETTING

The approach to dynamical systems and dynamical entropy used here can be found in [6] together with a number of concrete examples and references to original papers and alternative formalisms.

We assume that all bounded observables of our system generate a C^*-algebra $\mathcal{A}$ with unit $\mathbf{1}$. The discrete-time (reversible) dynamics is given in terms of an automorphism Θ acting on $\mathcal{A}$. By ω we denote a state on $\mathcal{A}$ invariant with respect to Θ. This state describes the reference (initial) state of the system, for instance: ground state (e.g. vacuum state of the electromagnetic field), thermal equilibrium state, nonequilibrium stationary state (e.g. stream of particles), etc. The triple $(\mathcal{A}, \Theta, \omega)$ will be called a *quantum dynamical system.* For infinite systems the $C^* - algebra$ $\mathcal{A}$ contains elements which do not correspond to observables measured by any finite apparatus but rather describe properly defined limits of physical observables. Therefore, it is necessary to consider a subalgebra $\mathcal{A}_0$ of physically admisible observables called *local* or *smooth* subalgebra. The local algebra should be invariant with respect to dynamics i.e. for $A \in \mathcal{A}_0$, $\Theta(A) \in \mathcal{A}_0$ too.

Ergodic properties of quantum dynamical systems are usually defined in terms of system's reaction to external perturbations. Any such perturbation can be realised by a *completely positive unity preserving map* $\Lambda : \mathcal{A} \mapsto \mathcal{A}$. We restrict ourselves to local and finite perturbations given by the formula

$$\Lambda_{\mathbf{X}}(A) = \sum_{j=1}^{k} X_j^* A X_j \tag{1}$$

where $\mathbf{X} = \{X_1, X_2 ..., X_k; X_j \in \mathcal{A}_0; \sum_{j=1}^{k} X_j^* X_j = \mathbf{1}\}$ is an *operational partition of unity.*

A completely positive map (1) perturbs the reference state ω yielding a new perturbed one which is defined in terms of the mean values

$$\omega^{\mathbf{X}}(A) = \sum_{j=1}^{k} \omega(X_j^* A X_j) \ , \ A \in \mathcal{A} \ . \tag{2}$$

One should notice that different partitions can produce the same completely positive map. For two partitions $\mathbf{X}, \mathbf{Y}$ we define a finer partition $\mathbf{X} \circ \mathbf{Y} = \{X_j Y_l; j = 1, 2, ..., k, l = 1, 2, ..., r\}$ and the corresponding completely positive map $\Lambda_{\mathbf{X} \circ \mathbf{Y}} = \Lambda_{\mathbf{Y}} \Lambda_{\mathbf{X}}$. Both, the set of partitions $\mathcal{P}(\mathcal{A}_0)$ and the set of corresponding completely positive maps $\mathcal{M}(\mathcal{A}_0)$ form semigroups with respect to compositions and with a unity given by a trivial partition $\mathbf{I} = \{\mathbf{1}\}$. There are important subsemigroups of partitions and maps :

a)$\mathcal{P}^b(\mathcal{A}_0)$ and $\mathcal{M}^b(\mathcal{A}_0)$ generated by bistochastic partitions i.e. $\sum_{j=1}^{k} X_j X_j^* = \mathbf{1}$

b)$\mathcal{P}^u(\mathcal{A}_0)$ $\mathcal{M}^u(\mathcal{A}_0)$ generated by unitary partitions i.e. $X_j^* X_j = X_j X_j^* = \mu_j \mathbf{1}; j = 1, 2, ..., k$.

Obviously, $\mathcal{P}^u(\mathcal{A}_0) \subset \mathcal{P}^b(\mathcal{A}_0) \subset \mathcal{P}(\mathcal{A}_0)$ and $\mathcal{M}^u(\mathcal{A}_0) \subset \mathcal{M}^b(\mathcal{A}_0) \subset \mathcal{M}(\mathcal{A}_0)$. The different subsemigroups correspond to the different means used to perturb our system. Unitary partitions can be realized by external classical possibly random "potentials", bistochastic ones can involve interaction with quantum environment at infinite temperature (tracial) state, which displays some classical features, while general partitions need generic quantum ancillary resources. It is important that the bistochastic maps does not decrease the entropy of the system.

A. Hilbert space representation

It is often very convenient to use a canonical representation of a dynamical system $(\mathcal{A}, \Theta, \omega)$ in terms of:

a) the Hilbert space $\mathcal{H}_\omega$,

b) the representation of the algebra $\mathcal{A}$ in the algebra of bounded operators $B(\mathcal{H}_\omega)$ i.e. any element A of $\mathcal{A}$ is represented by an operator $\hat{A}$ and the map $A \mapsto \hat{A}$ preserves the algebraic structure,

c) the state ω is represented by the normalized Hilbert space vector $|\Omega> \in \mathcal{H}_\omega$ such that $\omega(A) =< \Omega, \hat{A}\Omega >$,

d) the dynamical automorphism Θ is represented by the unitary operator $\mathcal{U}$, $\mathcal{U}|\Omega >= |\Omega >$ and for $B = \Theta(A)$, $\hat{B} = \mathcal{U}^* \hat{A} \mathcal{U} \equiv \hat{\Theta}(\hat{A})$. It is useful to define the Schrödinger picture of the dynamics in the Hilbert space representation by a transposed map

$$\hat{\Theta}^T(\hat{\rho}) = \mathcal{U} \hat{\rho} \mathcal{U}^* \ . \tag{3}$$

The Hilbert space $\mathcal{H}_\omega$ can be identified with the algebra $\mathcal{A}$ equipped with the scalar product $< A, B > = \omega(A^* B)$ modulo the equivalence relation: $A \equiv B$ if and only if $\omega[(A - B)^*(A - B)] = 0$. The element defined by the unity $\mathbf{1}$ in $\mathcal{A}$ is exactly our normalized vector Ω. Any element A of the algebra $\mathcal{A}$ is represented by the operator $\hat{A}$ which is defined by the left multiplication. Operators corresponding to right multiplication form an algebra of observables of a "minimal environment" (ancilla).

The formalism of above, called in the mathematical literature GNS representation, has been rediscovered by physicists under the names of the Liouville space approach [7], thermofield formalism [8] or in the context of quantum information theory as "state purification by ancilla" [2]. Its physical meaning for finite systems is clear: for a system being in a mixed state we reconstruct its minimal dilation described by a pure entangled state which reproduces the original state as a reduced marginal one.

Completely positive maps discussed above act in GNS representation on the whole operator algebra $B(\mathcal{H}_\omega)$

$$\hat{\Lambda}_{\mathbf{X}}(B) = \sum_{j=1}^{k} \hat{X}_j^* B \hat{X}_j \ , \ B \in B(\mathcal{H}_\omega) \tag{4}$$

and the perturbed state $\omega^{\mathbf{X}}$ is represented by the density matrix

$$\hat{\rho}[\mathbf{X}] = \hat{\Lambda}_{\mathbf{X}}^T(|\Omega >< \Omega|) = \sum_{j=1}^{k} |\hat{X}_j \Omega >< \hat{X}_j \Omega| \tag{5}$$

where $\hat{\Lambda}_{\mathbf{X}}^T$ is the (GNS) Schrödinger picture version of the Heisenberg picture map $\hat{\Lambda}_{\mathbf{X}}$.

B. Quantum dynamical entropy

We briefly sketch the theory of quantum dynamical entropy defined in terms of operational partitions of unity. The basic object in this approach is the following multi-time correlation matrix generated by the partition $\mathbf{X} = \{X_1, X_2, ..., X_k\}$

$$\rho[\mathbf{X}^n]_{i_1,...,i_n; j_1,...,j_n} =$$

$$\omega\big(X_{j_1}^* \Theta(X_{j_2}^*) ... \Theta^{n-1}(X_{j_n}^*) \Theta^{n-1}(X_{i_n}) ... \Theta(X_{i_2}) X_{i_1}\big) \ . \tag{6}$$

$\rho[\mathbf{X}^n]$ is a positively defined, $k^n \times k^n$ complex-valued matrix with a trace equal one. Therefore, the sequence $\{\rho[\mathbf{X}^n]; n = 1, 2, 3, ...\}$ can be treated as a consistent family of reduced density matrices which describes the state of a one-sided chain of quantum "spins". To any spin at a given site corresponds a k-dimensional Hilbert space and $\rho[\mathbf{X}^n]$ is a mixed state of n spins located at the sites $\{0, 1, ..., n-1\}$. Then a single step of the evolution translates into the right shift on the spin chain and we obtain a *quantum symbolic dynamics* .

The von Neumann entropy of the density matrix $\rho[\mathbf{X}^n]$ measures an amount of information encoded in the multitime correlations:

$$S(\rho[\mathbf{X}^n]) = -\mathrm{tr}\big(\rho[\mathbf{X}^n] \ln \rho[\mathbf{X}^n]\big) \tag{7}$$

The entropy of the partition $h[\omega, \Theta, \mathbf{X}]$ is defined as a limit

$$h[\omega, \Theta, \mathbf{X}] = \limsup_{n\to\infty} \frac{1}{n} S(\rho[\mathbf{X}^n]) \ . \tag{8}$$

Finally, the dynamical entropy of Θ is a supremum over all physically admisible (local) partitions

$$h[\omega, \Theta, \mathcal{A}_0] = \sup_{\mathbf{X}\in\mathcal{P}(\mathcal{A}_0)} h[\omega, \Theta, \mathbf{X}] \ . \tag{9}$$

Restricting the supremum to subsemigroups $\mathcal{P}^u(\mathcal{A}_0)$ or $\mathcal{P}^b(\mathcal{A}_0)$ we obtain corresponding restricted dynamical entropies satisfying the obvious inequality

$$h^u[\omega, \Theta, \mathcal{A}_0] \leq h^b[\omega, \Theta, \mathcal{A}_0] \leq h[\omega, \Theta, \mathcal{A}_0] \ . \tag{10}$$

The equivalent expression for (8) can be obtained in the GNS representation

$$S(\rho[\mathbf{X}^n]) = S(\hat{\rho}[\mathbf{X}^n]) \tag{11}$$

where (see eqs (5)(6))

$$\hat{\rho}[\mathbf{X}^n] = [\hat{\Theta}^T \hat{\Lambda}^T_{\mathbf{X}}]^n (|\Omega><\Omega|) \ . \tag{12}$$

The formula of above suggests a new interpretation of $S(\rho[\mathbf{X}^n])$ as the entropy of the density matrix obtained by repeated measurements performed on the evolving system + ancilla.

It has been proved that for classical systems the scheme of above reproduces the standard Kolmogorov-Sinai entropy, and a for a number of infinite quantum systems the dynamical entropy (9) has been computed. Moreover, in the known examples all three entropies (10) coincide. Although, strictly speaking, the dynamical entropy exists only for classical or infinite quantum systems the n-dependence of the entropy $S(\rho[\mathbf{X}^n])$ provides interesting informations about "quantum chaos" in finite quantum systems as well.

There exist other, nonequivalent definitions of quantum dynamical entropy among them CNT-entropy is the most developed one [9]. Preliminary results on its information-theoretical meaning can be found in [10].

C. Quantum Bernoulli shifts

The simplest example of infinite quantum dynamical system is a quantum Bernoulli shift. Consider an infinite collection of the identical quantum systems ("spins") attached to the sites of 1-dimensional lattice labeled by the integers $\mathbf{Z}$. The single site algebra is a $d \times d$ matrix algebra $\mathbf{M}_d$ and $\mathcal{A}_{[-n,n]}$ denotes the algebra localized on $[-n, n]$ and given by a suitable tensor product of $\mathbf{M}_d$. The local algebra of observables $\mathcal{A}_0 = \bigcup_{n\in\mathbf{N}} \mathcal{A}_{[-n,n]}$ can be completed to a C^*- algebra $\mathcal{A}$ of *quasi-local observables*. The discrete time dynamics Θ is given by a shift to the right which is an automorphism on $\mathcal{A}$ leaving $\mathcal{A}_0$ invariant. The state ω of the considered system is a product state $\otimes_{\mathbf{Z}}\rho$ where ρ is a single-site state given by a $d\times d$ density matrix. Obviously, ω is shift invariant.

One can easily compute the dynamical entropies (9)(10) for the quantum Bernoulli shift which are equal

$$h^u[\omega, \Theta, \mathcal{A}_0] = h^b[\omega, \Theta, \mathcal{A}_0] = h[\omega, \Theta, \mathcal{A}_0] = S(\rho) + \ln d \ . \tag{13}$$

To prove it one can notice that the RHS of eq.(13) is an upper bound for any $h[\omega, \Theta, \mathbf{X}]$, $\mathbf{X} \in \mathcal{P}(\mathcal{A}_0)$ due to a general inequality

$$S(\sigma[\mathbf{X}]) \leq S(\sigma) + \ln N \tag{14}$$

where σ is a density matrix on an N- dimensional Hilbert space, $\mathbf{X}$ is an arbitrary partition of unity and $\sigma[\mathbf{X}]_{ij} = \mathrm{tr}(\sigma X_j^* X_i)$. This bound is reached for a local, single-site unitary partition

$$\mathbf{W} = \{d^{-1}W(k, l); k, l = 1, 2, ..., d\} \tag{15}$$

where $W(k, l)$ are *discrete Weyl operators* defined in terms of the basis $\{|e_m>; m = 1, 2, ..., d\}$ of eigenvectors of ρ by the formula

$$W(k, l)|e_m>= \exp(i2\pi k/d)|e_{m\oplus l}> \ ,$$

$$m \oplus l = m + l \ (\mathrm{mod}\ d) \ . \tag{16}$$

III. COMMUNICATION CHANNEL WITH CLASSICAL INPUT AND OUTPUT

We consider a model of communication channel for which the input and output are strings of letters and the physical carrier of information is a quantum dynamical system described in the C^*-algebraic language by $(\mathcal{A}, \mathcal{A}_0, \Theta, \omega)$ as in the previous Section.

A. Input and output

A given input message of the length n is a sequence $\alpha_1, \alpha_2, ...\alpha_n$ of letters which belong to a certain alphabet identified with $\{1, 2, ..., r\}$. Any letter α is transmitted by means of a perturbation of the reference state ω by a completely positive map $\Lambda_\alpha \in \mathcal{M}(\mathcal{A}_0)$. This encoding procedure will be shortly denoted by $\mathbf{\Lambda}$. We can restrict possible perturbations to entropy increasing ones i.e. $\Lambda_\alpha \in \mathcal{M}^u(\mathcal{A}_0)$ or $\Lambda_\alpha \in \mathcal{M}^b(\mathcal{A}_0)$. Two consecutive perturbations of the state ω are always separated by the action of the dynamics Θ. This can be regarded as a definition of a letter which is the basic unit of the message sent during the single evolution step (unit of time). Therefore for a n-letter message we have the corresponding completely positive perturbation

$$\bar{\alpha} \equiv (\alpha_1, \alpha_2, ...\alpha_n) \mapsto \Lambda_{\alpha_1}\Theta\Lambda_{\alpha_2}\Theta \cdots \Lambda_{\alpha_n}\Theta \ . \tag{17}$$

It is convenient to use the Hilbert space (GNS) representation to associate with a given massage $\bar{\alpha} = (\alpha_1, \alpha_2, ...\alpha_n)$ a density matrix $\hat{\rho}(\bar{\alpha})$ acting on the Hilbert space $\mathcal{H}_\omega$ which can be written using the notation (6)(12)(17) as

$$\hat{\rho}(\bar{\alpha}) = \hat{\Theta}^T\hat{\Lambda}^T_{\alpha_n} \cdots \hat{\Theta}^T\hat{\Lambda}^T_{\alpha_2}\hat{\Theta}^T\hat{\Lambda}_{\alpha_1}(|\Omega><\Omega|) \ . \tag{18}$$

Receiving of a message is realized by performing a measurement of the suitable decoding observable $\mathbf{D}$ with possible outcomes $(\delta_1, \delta_2, ...\delta_m)$. Here $\mathbf{D} = \{D_1, D_2, ...D_m; D_k \in B(\mathcal{H}_\omega), D_k \geq 0, \sum_{k=1}^m D_k = \mathbf{1}\}$ is a *generalized observable* (or "fuzzy observable"). Choosing D_k from the whole $B(\mathcal{H}_\omega)$ means that we are able to exctract the information encoded in entanglement of the system with its environment. If we can perform the measurements on the dynamical system only we put $D_k \in \mathcal{A}$ identifying any element of $\mathcal{A}$ with its operator representation in $B(\mathcal{H}_\omega)$.

The basic quantity is the conditional output probability $P(\bar{\alpha}|\delta_j)$ which gives the probability of recording the output δ_j under the condition of the input message $\bar{\alpha}$ [9]

$$P(\bar{\alpha}|\delta_j) = \mathrm{tr}(\hat{\rho}(\bar{\alpha})D_j) \ . \tag{19}$$

Having a given input probability distribution $p_{in} = \{p_{in}(\bar{\alpha})\}$ we can define the output probability distribution

$$p_{out}(\delta_j) = \sum_{\bar{\alpha}} p_{in}(\bar{\alpha})P(\bar{\alpha}|\delta_j) \tag{20}$$

and the input-output probability distribution

$$p_{in,out}(\bar{\alpha}, \delta_j) = p_{in}(\bar{\alpha})P(\bar{\alpha}|\delta_j) \ . \tag{21}$$

The standard definition of the *amount of transmitted information* is given in terms of Shannon entropies $S(p) = -\sum p_k \ln p_k$ [11]

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) = S(p_{in}) + S(p_{out}) - S(p_{in,out})$$

$$= S(p_{out}) - \sum_{\bar{\alpha}} p_{in}(\alpha)S(P(\bar{\alpha}|\cdot)) \tag{22}$$

and satisfies the following inequalities

$$0 \leq I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq \min\{S(p_{in}), S(p_{out})\} \ . \tag{23}$$

The Holevo-Levitin inequality [11] provides an upper bound on $I(p_{in}, \mathbf{\Lambda}, \mathbf{D})$ which is independent of the choice of an output device

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq S\Big(\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\hat{\rho}(\bar{\alpha})\Big) - \sum_{\bar{\alpha}} p_{in}(\bar{\alpha})S(\hat{\rho}(\bar{\alpha})) \ . \tag{24}$$

B. Channel capacities

The important quantity which characterizes the efficiency of a communication channel is its capacity. In our case it will be an averaged amount of classical information, transmitted per unit of time, maximized over definite sets of information sources, encoding and decoding procedures and calculated in the limit of infinitely long input messages

$$\mathcal{C} = \sup_{\{p_{in}\},\{\mathbf{\Lambda}\},\{\mathbf{D}\}} \Big\{\limsup_{n\to\infty} \frac{1}{n} I(p_{in}, \mathbf{\Lambda}, \mathbf{D})\Big\} \ . \tag{25}$$

In the following we shall discuss several cases of capacity:

a)The entanglement-assisted classical capacity C_E [12] which corresponds to the supremum taken over all information sources, arbitrary encoding procedures $\mathbf{\Lambda} \subset \mathcal{M}(\mathcal{A}_0)$ and arbitrary decoding observables $\mathbf{D} \subset B(\mathcal{H}_\omega)$.

b)The ordinary classical capacity C and its rectricted versions C_u, C_b corresponding to the supremum over all information sources, arbitrary decoding observables of the system alone, i.e. $\mathbf{D} \subset \mathcal{A}$, and encoding procedures involving completely positive perturbations from $\mathcal{M}(\mathcal{A}_0)$, $\mathcal{M}^u(\mathcal{A}_0)$ and $\mathcal{M}^b(\mathcal{A}_0)$ respectively.

c)The capacities of above restricted to Bernoulli sources, i.e. the information sources with product probability measures

$$p_{in}(\alpha_1, \alpha_2, ..., \alpha_n) = p(\alpha_1)p(\alpha_2)...p(\alpha_n) \tag{26}$$

and denoted by C^0_E, C^0, C^0_u, C^0_b.

The definitions of above imply obvious inequalities

$$C^0_u \leq C^0_b \leq C^0 \leq C^0_E \ , \ C_u \leq C_b \leq C \leq C_E,$$

$$C^0_E \leq C_E \ , \ C^0_u \leq C_u \ , \ C^0_b \leq C_b \ , \ C^0 \leq C \ . \tag{27}$$

The dynamics Θ of the system is reversible and therefore noise is not explicitly present in this scheme. There are several possibilities to introduce noise in our setting. The first, natural one, seems to be replacing an authomorphism Θ by a completely positive dynamical map. However, this would produce capacities typically equal to zero because the errors accumulate with a number of time steps except the situation where a proper scaling of noise with n is introduced. Another possibility consists in putting extra conditions on decoding observables $\mathbf{D}$, assuming that $\mathbf{D} \subset \mathcal{B}$ where $\mathcal{B}$ is a proper subalgebra of $\mathcal{A}$ or $B(\mathcal{H}_\omega)$. A certain type of background noise appears in the case of capacities C_u, C_b or C_u^0, C_b^0 due to a mixed reference state ω which cannot be locally purified by entropy increasing perturbations (see Section IV).

C. Dynamical entropy bound

We prove our first result which provides the relation between ergodic properties of the channel treated as a dynamical systems and its entanglement-assisted capacity for the case of Bernoulli sources.

Theorem 1 For any quantum dynamical system

$$C_E^0 \leq h[\omega, \Theta, \mathcal{A}_0] \ . \tag{28}$$

The proof follows from the Holevo-Levitin inequality (24) and the definitions (8)(9). For a Bernoulli source and a given encoding $\mathbf{\Lambda}$ there exists a partition of unity $\mathbf{X}$ such that $\Lambda_{\mathbf{X}} = \sum_\alpha p(\alpha)\Lambda_\alpha$ Therefore, for any message of length n (see eqs(12)(18))

$$\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\hat{\rho}(\bar{\alpha}) = \hat{\rho}[\mathbf{X}^n] = [\hat{\Theta}^T \hat{\Lambda}^T_{\mathbf{X}}]^n (|\Omega><\Omega|) \tag{29}$$

and applying (9)(24)

$$\limsup_{n\to\infty} \frac{1}{n} I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq h[\omega, \Theta, \mathbf{X}] \leq h[\omega, \Theta, \mathcal{A}_0] \ . \tag{30}$$

The natural questions arise, how tight is this bound and whether it is possible to prove it for more general sources. This will be discussed in the next Section for quantum Bernoulli shifts.

IV. CAPACITIES FOR QUANTUM BERNOULLI SHIFTS

Perturbations of the reference state for quantum Bernoulli shifts propagate in a very simple way what allows to prove much stronger results than those given by (27)(28).

Theorem 2 For a quantum Bernoulli shift the following equalities hold

$$C_u^0 = C_b^0 = C_u = C_b = \ln d - S(\rho) \ , \tag{31}$$

$$C^0 = C = \ln d \ , \tag{32}$$

$$C_E^0 = C_E = \ln d + S(\rho) \ . \tag{33}$$

The interpretation of these results is quite obvious. The nonzero single-site entropy $S(\rho)$ can be an obstacle (noise) or an asset depending on the control we have of the system and its environment. Assume first, that we have no acces to environment. Then, if we can use entropy increasing perturbations only, $S(\rho)$ is an amount of noise which reduces the capacity of the channel. Applying arbitrary encoding with the help of ancillary resources we can reach a capacity $\ln d$.

On the other hand, if we can control the environment, represented here as an ancillary spin chain with the prior entanglement for any pair spin-ancilla, $S(\rho)$ becomes an amount of entaglement per site which improves the capacity. This is exactly the idea of *quantum dense coding* [2]. Moreover, C_E^0 reaches its upper bound (28) and the Bernoulli sources are optimal for all studied examples of capacities. One can expect that the bound (28) is tight and the Bernoulli sources are optimal for a larger class of quantum dynamical systems at reference states satisfying certain clustering properties with respect to dynamics. Finally, one should notice that for the quantum Bernoulli shift the CNT-entropy is equal to $S(\rho)$.

A. Proof of Theorem 2

In the first part of the proof we use again Holevo-Levitin inequality (24)and the fact that all perturbations of the state are strictly local and their propagation is given simply by a shift. Hence, for a given encoding $\mathbf{\Lambda}$ all completely positive maps are localised on the sites in a certain interval $[-l, l]$. After n time steps the total perturbation is localised in $[-l, l+n]$ and the perturbed state on the quasilocal algebra $\bigotimes_{\mathbf{Z}} M_d$ can be replaced by a local density matrix

$$\rho(\bar{\alpha}) = \Lambda^T(\bar{\alpha})(\otimes_{[-l,l+n]}\rho) \tag{34}$$

where $\Lambda^T(\bar{\alpha})$ is a total perturbation map in Schrödinger picture (not to be confused with GNS representation (18)!). Applying now inequality (24) for the spin system living on the interval $[-l, l+n]$

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq S\Big(\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\rho(\bar{\alpha})\Big) - \sum_{\bar{\alpha}} p_{in}(\bar{\alpha}) S(\rho(\bar{\alpha})) \tag{35}$$

we obtain for arbitrary perturbations

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq (n + 2l + 1)\ln d \tag{36}$$

while for the entropy increasing ones

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq (n + 2l + 1)(\ln d - S(\rho)) \ . \tag{37}$$

To obtain a bound useful for C_E we use a GNS representation and the bound

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq S\Big(\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\hat{\rho}(\bar{\alpha})\Big) \ . \tag{38}$$

For any n there exists a partition of unity $\mathbf{Y}_n$ which is generally not a composition of n partitions like $\mathbf{X}^n$ in (6) but nevertheless is localised on the interval $[-l, l+n]$ such that

$$\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\hat{\rho}(\bar{\alpha}) = \hat{\rho}[\mathbf{Y}_n] \ . \tag{39}$$

Then using (24) and the general bound (14)

$$I(p_{in}, \mathbf{\Lambda}, \mathbf{D}) \leq S\Big(\sum_{\bar{\alpha}} p_{in}(\bar{\alpha})\hat{\rho}(\bar{\alpha})\Big)$$

$$= S(\hat{\rho}[\mathbf{Y}_n]) = S(\rho[\mathbf{Y}_n]) \leq (n+2l+1)(S(\rho)+\ln d) \ . \tag{40}$$

The proof of the upper bounds is completed by dividing both sides of (36)(37)(40) by n , taking limit $n \to \infty$ and proper suprema over $p_{in}, \mathbf{\Lambda}$ and $\mathbf{D}$.

In the second part of the proof we show that the upper bounds are reached choosing proper Bernoulli sources, single-site encoding perturbations and suitable decoding observables. In this case $p_{in}(\bar{\alpha}) = p(\alpha_1)p(\alpha_2)\cdots p(\alpha_n)$ and $\rho(\bar{\alpha}) = \rho(\alpha_1)\rho(\alpha_2)\cdots\rho(\alpha_n)$ what is exactly the setting of the Holevo-Schumacher-Westmoreland theorem [13] which may be formulated as follows.

Theorem 3 Take a Bernoulli source and a single-site encoding as above. Then, by a suitable choice of a decoding observable the asymptotic amount of transmitted information per unit of time can be arbitrarily close to the Holevo-Levitin bound

$$S\big(\sum_{\alpha} p(\alpha)\rho(\alpha)\big) - \sum_{\alpha} p(\alpha)S\big(\rho(\alpha)\big) \ . \tag{41}$$

It remains to compute the bound (41) for different schemes corresponding to the capacities C^0, C_u^0 and C_E^0 respectively.

For C^0 we take d letters with a priori probabilities $1/d$ and the single-site perturbations

$$\Lambda_{\alpha}^T(\sigma) = \mathrm{tr}(\sigma)|e_{\alpha} >< e_{\alpha}| \ , \ \sigma \in \mathbf{M}_d \tag{42}$$

where $\{|e_{\alpha>}\}$ is a basis for a single spin. The bound (41) is obviously equal to $\ln d$.

For C_u^0 we take equally distributed d^2 letters with unitary single-site encoding given by the discrete Weyl operators $W(l,k)$ (15). Then using the fact that for any single spin matrix σ

$$\frac{1}{d^2}\sum_{k,l=1}^{d} W(k,l)\sigma W(k,l)^* = \mathrm{tr}(\sigma)\frac{1}{d}\mathbf{1} \tag{43}$$

we obtain the bound (41) equal to $\ln d - S(\rho)$.

To reach the bound for C_E^0 we consider a purification of Bernoulli shift with a pure single-site reference state of spin - ancilla

$$\tilde{\rho} = \sum_{j=1}^{d} \sqrt{\lambda}_j |e_j > \otimes |e'_j > \tag{44}$$

being a purification of $\rho = \sum_{j=1}^{d} \lambda_j |e_j >< e_j|$. Taking again equally distributed d^2 letters with unitary single-site encoding given by the unitary operators $W(l,k) \otimes \mathbf{1}$ we reach the bound $\ln d + S(\rho)$.

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